LNCT GROUP OF COLLEGES

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Department: Electronics & Comminication

Subject: Digital Signal Processing EC601

Unit:III

Topic:Discrete Fourier Transform

IIR:- have zeros & poles

FIR:- have only zeros.

+ Discrete fourier Transform :-

If $x(n)$ is DTS, its N pt DFT is given as-

$$X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N} \cdot n \cdot k}$$

where $K = 0, 1, 2, \ldots, N-1$.

and N pt IDFT is given as-

$$x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K)\, e^{j\frac{2\pi}{N} \cdot n \cdot k}$$

where $n = 0, 1, \ldots, N-1$

$$\boxed{e^{-j\frac{2\pi}{N}} = \omega_N.}$$

It is k/a. "TWIDDLE FACTOR" or PHASE FACTOR.

In terms of twiddle factor.

DFT is -

$$X(k) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{nk}.$$

where, $k = 0, 1, \ldots N-1$

and IDFT is -

$$x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(k)\, W_N^{-nk}$$

Q. Determine two pt DFT of sequence

$x(n) = \{1, -1\}$

Here $N = 2$.

$$X(K) = \sum_{n=0}^{1} x(n)\, e^{-j\frac{2\pi}{N} \cdot n \cdot k}.$$

and $K = 0, 1$.

$$\cancel{\therefore X(K) = 1 \cdot e^{-j\frac{2\pi}{2} \cdot n \cdot 0} + (-1) e^{-j\frac{2\pi}{2} \cdot n \cdot 1}}$$

$$\cancel{= 1 - e^{-j\pi n}.}$$

$$X(0) = x(0) \cdot e^{-j\pi \cdot 0} + x(1)\, e^{-0}.$$
$$= 1 + (-1) = 0.$$

$X(0) = 0$

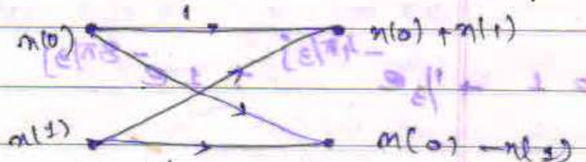
$$X(1) = x(0) \cdot e^{-j2\pi/2 \cdot 0 \cdot 1} + x(1) \cdot e^{-j2\pi/2 \cdot 1 \cdot 1}$$
$$= 1 - e^{-j\pi}$$
$$= 1 - (\cos\pi - j\sin\pi)$$

$X(1) = 2$

$\therefore X(0) = 0$ for $K = 0$.
$X(1) = 2$ for $K = 1$.

$X(K) = \{0, 2\}$

$\angle X(K) = \{0, 0\}$



Two pt DFT
1st sample ⇒ Sum of n(0) & n(1)
2nd sample ⇒ diff. b/w n(0) & n(1)

Qo. Determine three point DFT of $x(n) = \{1, 1/3, 1\}$.

Here, $N = 3$, $K = 0, 1, 2$.

$$\therefore X(k) = \sum_{n=0}^{2} x(n) \cdot e^{-j\frac{2\pi}{N} \cdot n \cdot k}$$

$$X(0) = x(0) \cdot e^{-j\frac{2\pi}{3}\cdot 0 \cdot 0} + x(1) \cdot e^{-j\frac{2\pi}{3}\cdot 1 \cdot 0} + x(2) \cdot e^{-j\frac{2\pi}{3}\cdot 2 \cdot 0}$$

$$= 1 \cdot e^0 + 1/3 e^0 + 1 \cdot e^0$$

$$X(0) = 2 + 1/3 = 7/3.$$

$$X(1) = x(0) \cdot e^{-j 2\pi/3 \cdot 1 \cdot 0} + x(1) \cdot e^{-j\frac{2\pi}{3}\cdot 1 \cdot 1} + x(2) \cdot e^{-j\frac{2\pi}{3}\cdot 1 \cdot 2}$$

$$= 1 \cdot e^0 + 1/3 \cdot e^{-j2\pi/3} + 1 e^{-j4\pi/3}$$

$$= 1 + 1/3 \left[\cos 2\pi/3 - j \sin 2\pi/3\right] + 1\left[\cos 4\pi/3 - j \sin 4\pi/3\right]$$

$$= 1 + 1/3\left[-1/2 - j\right.$$

$$X(1) = 4/3 + \sqrt{3}/3 j$$

$$X(2) = x(0) \cdot e^{-j\frac{2\pi}{3}\cdot 0 \cdot 2} + x(1) \cdot e^{-j\frac{2\pi}{3}\cdot 1 \cdot 2} + x(2) \cdot e^{-j2\pi/3 \cdot 2 \cdot 2}$$

$$= 1 + 1/3 e^{-4\pi/3 j} + 1 e^{-8\pi/3 j}$$

$$= 1 + \left[\cos \frac{4\pi}{3} - j \sin \frac{4\pi}{3}\right] 1/3 + 1\left[\cos \frac{8\pi}{3} - j \sin \frac{8\pi}{3}\right]$$

$$X(2) = 1 + \left[-1/2 + j\frac{\sqrt{3}}{2}\right] 1/3 + \left[-1/2 + j\sqrt{3}/2\right]$$

$$= 1 - 1/6 - \frac{\sqrt{3}}{3 \cdot 2} j - 1/2 + \frac{\sqrt{3}}{2} j$$

$$= \frac{6 - 1 - 3}{6} - \frac{\sqrt{3}}{2} j \left[\frac{1}{3} - 1\right]$$

$$= \frac{2}{6} - \frac{\sqrt{3}}{2} j \left[\frac{-2}{3}\right]$$

$$X(2) = 1/3 - \frac{\sqrt{3}}{3} j$$

$$\therefore X(0) = 7/3.$$

$$X(1) = 1/3 + j/\sqrt{3}$$

$$X(2) = 1/3 - 1/\sqrt{3} j$$

Three point DFT of $x(n) = \{1, 1/3, 1\}$ is –

$$X(0) = 7/3 \angle 0°$$

$$X(1) = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2} \angle \tan^{-1}\left(\frac{3}{\sqrt{3}}\right)$$

$$X(2) = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2} \angle \tan^{-1}\left(\frac{-3}{\sqrt{3}}\right)$$

Qo. Determine four point DFT of

$$x(n) = \{x(0), x(1), x(2), x(3)\}$$

$$x(n) = \cos n\pi/4.$$

$$\therefore x(n) = \left\{1, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right\}.$$

$\therefore X(K) = \sum_{n=0}^{N-1} x(n) \cdot e^{-j\frac{2\pi}{N} \cdot n \cdot K}$.

$N = 3, \quad K = 0, 1, 2, 3$.

$X(0) = x(0) \cdot e^{-j\frac{2\pi}{4} \cdot 0 \cdot 0} + x(1) \cdot e^{-j\frac{2\pi}{4} \cdot 1 \cdot 0} + x(2) \cdot e^{-j\frac{2\pi}{4} \cdot 2 \cdot 0} + x(3) \cdot e^{-j\frac{2\pi}{4} \cdot 3 \cdot 0}$

$= 1 e^0 + \frac{1}{\sqrt{2}} e^0 + 0 - \frac{1}{\sqrt{2}} e^0$

$X(0) = 1$.

$X(1) = x(0) \cdot e^{-j\frac{2\pi}{4} \cdot 1 \cdot 0} + x(1) \cdot e^{-j\frac{2\pi}{4} \cdot 1 \cdot 1} + x(2) \cdot e^{-j\frac{2\pi}{4} \cdot 2 \cdot 1} + x(3) \cdot e^{-\frac{2\pi}{4} \cdot 1 \cdot 3}$

$= 1 + \frac{1}{\sqrt{2}} \cdot e^{-\pi/2 j} + 0 - \frac{1}{\sqrt{2}} e^{-3\pi/4 j}$

$= 1 + \frac{1}{\sqrt{2}}\left[\cos\frac{\pi}{2} - j\sin\frac{\pi}{2}\right] - \frac{1}{\sqrt{2}}\left[\cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}\right]$

$= 1 + \frac{1}{\sqrt{2}}[0 - j] - \frac{1}{\sqrt{2}}[0 + j]$

$= 1 - \frac{1}{\sqrt{2}} j - \frac{1}{\sqrt{2}} j = 1 - \sqrt{2} j$

$X(1) = 1 - \sqrt{2} j$

$X(2) = x(0) \cdot e^{-j\frac{2\pi}{4} \cdot 2 \cdot 0} + x(1) e^{-j\frac{2\pi}{4} \cdot 2 \cdot 1} + x(2) e^{-j2\pi/4 \cdot 2 \cdot 2} + x(3) e^{-j2\pi/4 \cdot 2 \cdot 3}$

$= 1 + \frac{1}{\sqrt{2}} e^{-\pi j} - \frac{1}{\sqrt{2}} e^{-3\pi j}$

$= 1 + \frac{1}{\sqrt{2}}\left[\cos\pi - j\sin\pi\right] - \frac{1}{\sqrt{2}}\left[\cos 3\pi - j\sin 3\pi\right]$

$= 1 + \frac{1}{\sqrt{2}}[-1 - 0] - \frac{1}{\sqrt{2}}[-1 - 0]$

$X(2) = 1$

$X(3) = x(0) \cdot e^{-j\frac{2\pi}{4} \cdot 3 \cdot 0} + x(1) \cdot e^{-j\frac{2\pi}{4} \cdot 3 \cdot 1} + x(2) \cdot e^{-j\frac{2\pi}{4} \cdot 3 \cdot 2} + x(3) \cdot e^{-j\frac{2\pi}{4} \cdot 3 \cdot 3}$.

$= 1 + \frac{1}{\sqrt{2}} e^{-3/2\pi j} - \frac{1}{\sqrt{2}} e^{-9\pi/2 j}$

$= 1 + \frac{1}{\sqrt{2}}\left[\cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}\right] - \frac{1}{\sqrt{2}}\left[\cos\frac{9\pi}{2} - j\sin\frac{9\pi}{2}\right]$.

$X(2) = 1 + \sqrt{2} j$

$\therefore X(K) = \{1, 1 - \sqrt{2} j, 1, 1 + \sqrt{2} j\}$

Deduction -

$\therefore$ In terms of tooiddle factor.

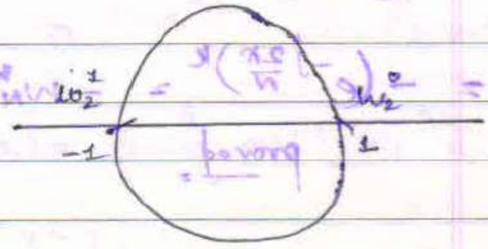
$X(K) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{n \cdot k}$.

$X(0) = x(0) + x(1) \ldots + x(3)$

$X(1) = x(0) + x(1) w^1 + x(2) w^2 + x(3) w^3$.

$X(2) = x(0) + x(1) w^2 + x(2) w^4 + x(3) w^6$.

$W_N = 1 \cdot e^{-j\frac{2\pi}{N}} = r e^{j\theta}$.



• properties of Twiddle factor :-

1. Periodicity :-

It is always periodic fun.

$$W_N^{(K+N)} = W_N^{K}.$$

$$W_N^{(K+rN)} = W_N^{K}.$$

proof :- let $W_N = e^{-j\frac{2\pi}{N}}$

$$\therefore \left(e^{-j\frac{2\pi}{N}}\right)^{(k+rN)} = e^{-j\frac{2\pi}{N}\cdot k} \cdot e^{-j\frac{2\pi}{N}\cdot N\cdot r}$$

$$= e^{-j\frac{2\pi}{N}\cdot k}\cdot \left(e^{-j2\pi}\right)^{r\cdot N}.$$

$$= e^{-j2\pi/N\, k} = W_N^{k}.$$

proved.

2. Half Periodicity :-

$$W_N^{(K+N/2)} = -W_N^{K}.$$

proof :- $W_N = e^{-j\frac{2\pi}{N}}$

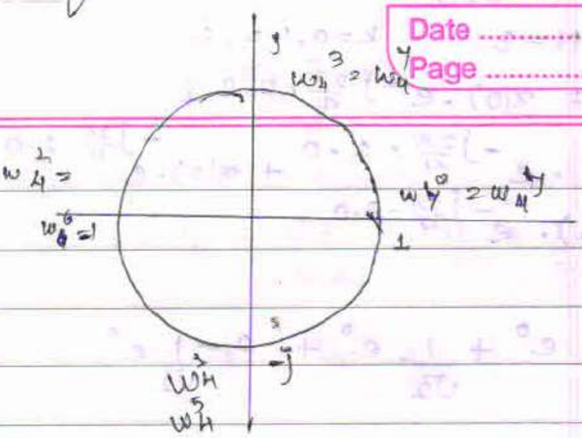
$$\therefore \left(e^{-j\frac{2\pi}{N}}\right)^{(k+N/2)} = \left(e^{-j\frac{2\pi}{N}}\right)^{k} \cdot \left(e^{-j\frac{2\pi}{N}}\right)^{N/2}.$$

$$= \left(e^{-j\frac{2\pi}{N}}\right)^{k}\cdot(-1)$$

$$= -\left(e^{-j\frac{2\pi}{N}}\right)^{k} = -W_N^{k}.$$

proved.

for N=4.



for half periodicity

$$W_4^{3} = W_4^{(1+4/2)} = -W_4^{1}$$

$$W_4^{2} = W_4^{(0+4/2)} = -W_4^{0}$$

3. $W_N^{N} = 1$; $W_N^{N/2} = -1$

Note:- Shortcut method.

let $x(n) = \left\{1, \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right\}$

$$X(k) = \sum_{n=0}^{3} x(n)\cdot W_4^{nk}.$$

$k = 0, 1, 2, 3$

$$X(0) = x(0) + x(1) + x(2) + x(3) = 1$$

$$X(1) = \sum_{n=0}^{3} x(n)\cdot W_4^{n}.$$

$$= x(0) + x(1)W_4^{1} + x(2)W_4^{2} + x(3)W_4^{3}.$$

$$= 1 + \frac{1}{\sqrt{2}}(-1) + 0\times(-1) + \left(-\frac{1}{\sqrt{2}}\right)j$$

$X(2) = x(0) + x(1)\cdot W_4^2 + x(2) W_4^4 + x(3)\cdot W_4^6$

$= 1 + \frac{1}{\sqrt{2}}(-1) + 0 + \left(\frac{-1}{\sqrt{2}}\right)(-1)$

$= 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 1$

$X(3) = x(0) + x(1) W_4^3 + x(2) W_4^6 + x(3)\cdot W_4^9$

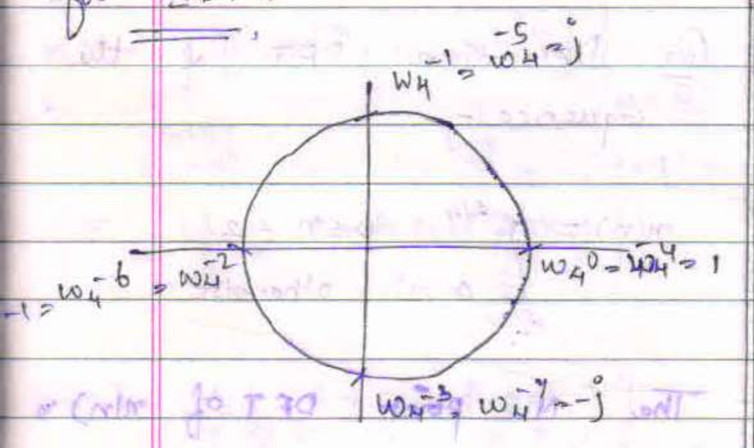
$= 1 + \frac{1}{\sqrt{2}}(j) + 0 + \left(\frac{-1}{\sqrt{2}}\right)(-j)$

$= 1 + \frac{j}{\sqrt{2}} + \frac{j}{\sqrt{2}} = 1 + \sqrt{2}j$

$X(K) = \{1,\ 1-\sqrt{2}j,\ 1,\ 1+\sqrt{2}j\}$

Note :- for DFT → clockwise
IDFT → anticlockwise

for IDFT



$x(n) = \frac{1}{N} \sum_{K=0}^{N-1} x(K)\cdot W_N^{-nK}$

$n = 0, 1, 2, \ldots$

$x(n) = \frac{1}{4} \sum_{K=0}^{3} X(K)\cdot W_N^{-nK}$

$x(0) = \frac{1}{4} \sum_{n=0}^{3} x(n) W_4^0$

$= \frac{1}{4}\left[x(0) + x(1) + x(2) + x(3)\right] 1$

$= \frac{1}{4}\left[1 + (1-\cancel{\sqrt{2}j}) + 1 + (1+\cancel{\sqrt{2}j})\right]$

$= \frac{8}{4}.$

$x(1) = \frac{1}{4} \sum_{k=0}^{3} x(k)\cdot W_4^{-k}$

$= \frac{1}{4}\left[X(0) W_4^{-0} + X(1)\cdot W_4^{-1} + X(2) W_4^{-2} + X(3) W_4^{-3}\right]$

$= \frac{1}{4}\left[1.1 + (1-\sqrt{2}j) j + 1.(-1) + (1+\sqrt{2}j).(-j)\right]$

$= \frac{1}{4}\left[\cancel{1} + \cancel{j} + \sqrt{2} - \cancel{1} - \cancel{j} + \sqrt{2}\right]$

$= \frac{2\sqrt{2}}{4} = \frac{\sqrt{2}}{2} = 1/\sqrt{2}.$

$x(2) = \frac{1}{4} \sum_{K=0}^{3} x(K)\cdot W_4^{-2k}.$

$= \frac{1}{4}\left[x(0)\cdot W_4^{-0} + X(1)\cdot W_4^{-2} + X(2)\cdot W_4^{-4} + X(3)\cdot W_4^{-6}\right]$

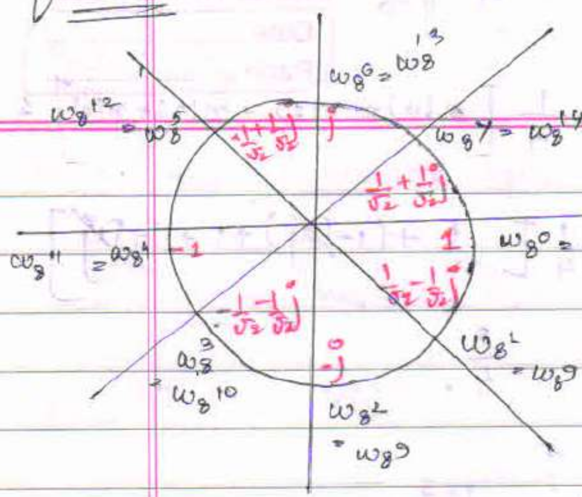
$= \frac{1}{4}\left[1.(1) + (1-\sqrt{2}j)(-1) + 1(-j) + (1+\sqrt{2}j)(-1)\right]$

$x(2) = 0$

$x(3) = \frac{1}{4} \sum_{k=0}^{3} x(K)\cdot W_4^{-3k}.$

$x(3) = 1/\sqrt{2}.$

Q: for N = 8



→ DFT and IDFT as Linear Transformation

Since, DFT ⇒

$$X(K) = \sum_{n=0}^{N-1} x(n).W_N^{nK}$$

$K = 0, 1, \ldots N-1$

$$X(0) = \sum_{n=0}^{N-1} x(n) = x(0) + x(1) + x(2) + \ldots$$

$$X(1) = x(0) + x(1).W_N^{1} + x(2)W_N^{2} + \ldots + x(N-1).W_N^{N-1}$$

$$\therefore X(N-1) = \sum_{n=0}^{N-1} x(n).W_N^{(N-1)n} =$$

$$x(0) + x(1).W_N^{(N-1)} + \ldots + x(N-1).W_N^{(N-1)(N-1)}$$

This can be expressed as matrix.

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \vdots \\ X(N-1) \end{bmatrix}_{N\times 1} = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & W_N^{1} & \ldots & & W_N^{N-1} \\ 1 & W_N^{2} & \ldots & & W_N^{2(N-1)} \\ \vdots & \vdots & \ldots & & \vdots \\ 1 & \ldots & & & W_N^{N(N-1)} \end{bmatrix}_{N\times N} \cdot \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{bmatrix}$$

Ex:-

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ \cancel{X(N-1)} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^{1} & W_4^{2} & W_4^{3} \\ 1 & W_4^{2} & W_4^{4} & W_4^{6} \\ 1 & W_4^{3} & W_4^{6} & W_4^{9} \end{bmatrix} \cdot \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \cdot \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

Q. Determine DFT of the sequence -

$$x(n) = \begin{cases} 1/4 & 0 \le n \le 2 \\ 0 & \text{otherwise} \end{cases}$$

The N-point DFT of x(n)

$$X(K) = \sum_{n=0}^{N-1} x(n)\, e^{j\frac{2\pi}{N}.nK}$$

$K = 0, 1, \ldots (N-1)$

$$x(n) = \{1/4, 1/4, 1/4\}$$

$\therefore\ X(K) = \frac{1}{4}\left[1 + e^{-j\omega} + e^{-j2\omega}\right]\Big|_{\omega = \frac{2\pi K}{N}}$

$= \frac{1}{4} e^{-j2\pi K/3}\left[1 + 2\cos\left(\frac{2\pi K}{3}\right)\right]$

Hence,

$X(K) = \frac{1}{4} \cdot e^{-j\frac{2\pi K}{3}}\left[1 + 2\cos\left(\frac{2\pi K}{3}\right)\right]$

where $K = 0, 1, \ldots N-1$

Qo. Determine DFT of sequence

$x(n) = \begin{cases} 1/5 & -1 \le n \le 1 \\ 0 & \text{otherwise} \end{cases}$

$x(n) = \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right)$

$X(K) = \sum_{n=0}^{\infty} x(n) e^{-j\omega n}$, at $\omega = \frac{2\pi k}{N}$.

$X(K) = \frac{1}{5}\left[e^{j\omega} + 1 + e^{-j\omega}\right]\Big|_{\omega = \frac{2\pi k}{N}}$

$= \frac{1}{5}\left[1 + 2\cos\left(\frac{2\pi k}{3}\right)\right]$

$X(K) = \frac{1}{5}\left[1 + 2\cos\left(\frac{2\pi k}{3}\right)\right]$ → ans

Qo Find N-point DFT for

$x(n) = a^n \qquad 0 < a < 1$

$X(K) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N} \cdot nk}, \quad K = 0, 1, \ldots (N-1)$

$X(K) = \sum_{n=0}^{N-1} a^n e^{-j\frac{2\pi}{N} nk}$

$= \frac{1 - \left(a e^{-j\frac{2\pi k}{N}}\right)^N}{1 - a e^{-j2\pi k/N}}$

$X(K) = \frac{1 - a^N}{1 - a e^{-j2\pi nk/N}}$. → ans

Qo. Determine IDFT of

$X(K) = \{3, (2+j), 1, (2-j)\}$

$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N} nk}$. $\quad 0 \le n \le N-1$

$N = 4, \quad x(n) = \frac{1}{4}\sum_{K=0}^{3} X(K) e^{j\frac{\pi nk}{2}}$

when $n=0$

$x(0) = \frac{1}{4}\sum_{k=0}^{3} X(k) e^{0}$

$= \frac{1}{4}\left[3 + (2+j) + 1 + (2-j)\right] = 2$

When $n = 1$.

$x(1) = \frac{1}{4}\left[3 + (2+j) e^{j\pi/2} + e^{j\pi} + (2-j) e^{j3\pi/2}\right]$.

$= \frac{1}{4}\left[3 + (2+j) j - 1 + (2-j)(-j)\right]$

$= 0$.

when $n = 2$.

$x(2) = \frac{1}{4}\left[3 + (2+j)(-1) + 1 + (2-j)(-1)\right] = 0$

Q. when $n=3$.

$$x(3) = \frac{1}{4}\sum_{k=0}^{3} X(k)\, e^{j3\pi k/2}.$$

$$x(3) = \frac{1}{4}\Big[3+(2+j)(-j)-1+(2-j)(j)\Big] = 1$$

$\therefore x(n) = \{2, 0, 0, 1\}$ → ans.

Imp.

- properties of DFT :-

1. Linearity
2. Periodicity
3. Circular shift of a Sequence
4. Time Reversal
5. Circular time shift.
6. Circular freq. shift.
7. Circular Convolution.
8. Complex Conjugation.
9. Multiplication of two Sequences
10. Parseval's theorem.
11. Symmetric property.

1. Linearity :-

If $x_1(n) \xleftrightarrow[DFT]{N} X(K) = \sum_{n=0}^{N-1} x_1(n) W_N^{nK}$,

$K = 0, 1, \dots, N-1$

$x_2(n) \xleftrightarrow[DFT]{N} X_2(K) = \sum_{n=0}^{N-1} x_2(n) W_N^{nK}$

$K = 0, 1, \dots, (N-1)$

then,

$x_3(K) = a_1 X_1(K) + a_2 X_2(K)$

$K = 0, 1, \dots, (N-1)$

proof :-

$$DFT[x_3(n)] = \sum_{n=0}^{(N-1)} x_3(n) W_N^{nK}$$

$K = 0, 1, \dots, (N-1)$

$$= \sum_{n=0}^{N-1} \big[a_1 x_1(n) + a_2 x_2(n)\big] W_N^{nK},$$

$K = 0, 1, \dots, (N-1)$

$$= a_1 X_1(K) + a_2 X_2(K).$$

proved.

2. Periodicity :-

If $x(n) \xleftrightarrow[DFT]{N} X(K)$

then, $X(K+N) = X(K)$

proof :- $X(K) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{nK}$.

$$X(K+N) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{n(K+N)}$$

$$= \sum_{n=0}^{N-1} x(n) \cdot W_N^{nK} \cdot W_N^{Nn}$$

Here $W_N^{Nn} = \left(e^{-j\frac{2\pi}{N}N}\right)^n$

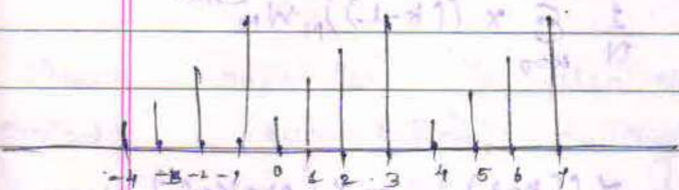
$= (e^{-j2\pi})^n = (1)^n = 1$

$$\therefore X(K+N) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{nK} = X(K)$$

proved.

3. Circular shift of a Sequence :-

In DFT we cannot combine linear shifted seq. with the original one due to loss of signal in the period.
So we do periodic shifting.



In periodic shifting - sequences can be (1 2 3 4) (4 1 2 3) (3 4 1 2) (2 3 4 1) (1 2 3 4)

$$\boxed{\underbrace{x_c(n)}_{\text{circular shift}} = \underbrace{x_p(n-K)}_{\text{periodic shifting}}}$$

K = no. of positions (shifted / shifting periodic)

ie $x_c(0) = x_p(0-1) = 4$

$x_c(1) = x_p(1-1) = 1$

$x_c(2) = x_p(2-1) = 2$

$x_c(3) = x_p(3-1) = 3$.

$x_c(n) = x_p(n-K)$

$= x((n-K))_N$

another way of representing periodic shifting.

N = periodicity.

$$\boxed{\begin{aligned} x_c(n) &= x(N+n-K) \\ &= x_p((n-K))_N \end{aligned}}$$

Eg. $K=1$ $N=4$

then.

$X_c(n) = X((N+n-K))$

$x_c(0) = x(4+0-1) = x(3)$

$x_c(1) = x(4+1-1) = x(4)$

$x_c(2) = x(4+2-1) = x(1)$

$x_c(3) = x(4+3-1) = x(2)$.

4. Time Reversal:-

If $x(n) \xleftrightarrow[DFT]{N} X(K)$,

then,

$x((-n))_N \xleftrightarrow[DFT]{N} X((-K))_N$

$x(N-n) \xleftrightarrow[DFT]{N} X(N-K)$.

Proof:- $DFT[x((-n))_N] = \sum_{n=0}^{N-1} x((-n))_N W_N^{nK}$.

$K = 0, 1, \ldots, (N-1)$ $\quad W_N^{-Nn}$

$x((-n))_N = x(N-n)$ $\quad W_N^{(K-N)n}$

$= \sum_{n=0}^{N-1} x(N-n) W_N^{nK}$ $\quad x((K-N))$

$K = 0, 1, \ldots, (N-1)$

let $N-n = t$.

$DFT[x((-n))_N] = \sum_{t=N}^{1} x(t) \cdot W_N^{(N-t)K}$

$= \sum_{t=N}^{1} x(t) \cdot W_N^{-tK}$

$DFT[x((-n))_N] = (\sum_{t=N}^{1} x(t) W_N^{-tK}$

$$DFT[x((-n))_N] = \sum_{t=N}^{1} x(t) W_N^{-tk}$$

$$= \sum_{t=N}^{1} x(t) W_N^{-tk}$$

$$W_N^{Nt} = \sum_{t=N}^{1} x(t) \cdot W_N^{(N-k)t}$$

(2) $$= X(N-K) = X((-K))_N$$

5. Circular time shift :-

$$x(n) \xleftrightarrow[DFT]{N} X(K)$$

then,

$$x((n-l)) \xleftrightarrow[DFT]{N} W_N^{lk} X(K)$$

proof :-

$$DFT[x((n-l))_N] = DFT[x(N+n-l)]$$

$$= \sum_{n=0}^{N-1} x(N+n-l) W_N^{nk}$$

$K = 0, 1, \ldots (N-1)$.

put $N+n-l = t$

then

$$DFT[x((n-l))_N] = \sum_{t=N-l}^{2N-l-1} x(t) \cdot W_N^{(t+l-N)k}$$

$$= \sum_{t=N-l}^{2N-l-1} x(t) W_N^{(tk)} \cdot W_N^{lk} \cdot W_N^{-NK}$$

$$= \sum_{t=N-l}^{2N-l-1} x(t) W_N^{tk} * W_N^{lk}$$

$$= W_N^{lk} X(K)$$

proved.

6. Circular freq. shift

If $x(n) \leftrightarrow X(K)$

then,

$$x(n) W_N^{-ln} \xleftrightarrow[N]{DFT} X((K-l))$$

proof :-

$$IDFT[X((K-l))_N] =$$

$$\frac{1}{N} \sum_{K=0}^{N-1} X((K-l))_N W_N^{-nK}$$

$$X((K-l))_N = X(N+K-l)$$

$$IDFT[X((K-l))_N] = \frac{1}{N} \sum_{K=0}^{N-1} X(N+K-l) W_N$$

put $N+K-l = t$

$$IDFT[X((K-l))_N] = \frac{1}{N} \sum_{N-l}^{2N-l-1} X(t) \cdot W_N$$

$$= \frac{1}{N} \sum_{N-l}^{2N-l-1} x(t) \cdot W_N^{-tn} \cdot W_N^{-ln} \cdot W_N$$

$$IDFT[X((K-l))_N] = W_N^{-ln} \frac{1}{N} \sum_{t=N-l}^{2N-l-1} X(t)$$

$$= (W_N^{-ln} \cdot x(n))$$

now taking DFT on both sides we get -

$$X((K-l))_N = DFT[W_N^{-ln} x(n)]$$

$$\therefore X((K-l)) = x(n) W_N^{-ln}.$$

proved.

4. Circular Convolution :-

If $x_1(n) \xleftrightarrow[DFT]{N} X_1(K)$

$x_2(n) \xleftrightarrow[DFT]{N} X_2(K)$

and $X_3(K) = X_1(K) \cdot X_2(K)$

then,

$x_3(n) = x_1(n) \circledast x_2(n)$

circular convolution is also called periodic convolution having length equal to length of convolved sequences.

proof :- $x_3(n) = IDFT[X_3(K)]$

$x_3(n) = \frac{1}{N} \sum_{K=0}^{N-1} X_3(K) W_N^{-nK}$

$n = 0, 1, \ldots (N-1)$

Since,

$X_3(K) = X_1(K) \cdot X_2(K)$

$\therefore x_3(n) = \frac{1}{N} \sum_{K=0}^{(N-1)} X_1(K) \cdot X_2(K) \cdot W_N^{-nK}$ —(i)

let

$X_1(K) = \sum_{m=0}^{(N-1)} x_1(m) \cdot W_N^{mK}$ —(ii)

$K = 0, 1, \ldots (N-1)$

$X_2(K) = \sum_{l=0}^{(N-1)} x_2(l) W_N^{lk}$ —(iii)

$K = 0, 1, \ldots, (N-1)$

now substituting the values from eq. (ii) & (iii) into eq. (i).

$\therefore x_3(n) = \frac{1}{N} \sum_{K=0}^{(N-1)} \left[\sum_{m=0}^{(N-1)} x_1(m) \cdot W_N^{+mk} \right] \cdot \left[\sum_{l=0}^{(N-1)} x_2(l) W_N^{lk} \right] \cdot W_N^{-nK}$ —(iv)

$= \frac{1}{N} \sum_{m=0}^{N-1} x_1(m) \cdot \sum_{l=0}^{N-1} x_2(l) \sum_{K=0}^{N-1} W_N^{(m+l-n)K}$

for $\sum_{K=0}^{(N-1)} W_N^{(m+l-n)k}$

suppose $W_N^{(m+l-n)} = a$.

$\therefore \sum_{k=0}^{(N-1)} a^k = \begin{cases} \frac{1-a^N}{1-a} & \text{when } a \neq 1 \\ N & a = 1 \end{cases}$

Here we see that on considering $\sum_{k=0}^{(N-1)} a^k = \frac{1-a^N}{1-a}$

then eq. (iv) becomes zero.

$\therefore$ we will consider the next case. ie $\sum_{k=0}^{N-1} a^k = N$, $(a=1)$

$\frac{1 - W_N^{(m+l-n)N}}{1 - W_N^{(m+l-n)}}$

for existance of circular conv.

$a = 1$.

ie $W_N^{(m+l-n)} = 1$,

when,

$m+l-n = qN$

$x_3(n) = \frac{1}{N} \sum_{M=0}^{N-1} x_1(M) \sum_{l=0}^{N-1} x_2(l) \cdot N$

$m+l-n = q$

$$x_3(n) = \sum_{m=0}^{N-1} x_1(m) \sum_{l=0}^{N-1} x_2(l)$$

$m + l - n = qN.$

$$m + l - n = qN$$

$$l = qN + n - m$$

$$l = ((n-m))_N$$

$$x_3(n) = \sum_{m=0}^{N-1} x_1(m) \cdot x_2((n-m))_N$$

(circular conv.)

$n = 0, 1, \ldots, (N-1)$

$$x_3(n) = \sum_{k=-\infty}^{\infty} x_1(k) \cdot x_2(n-k)$$

$$= x_1(n) * x_2(n)$$

(linear conv).

$$x_3(n) = x_1(n) \odot x_2(n).$$

proved.

Methods of Circular Convolution :-

- Tabular
- Using Circular array.
- using DFT & IDFT (Stockham's Method)
- Matrix

Q. Determine $\odot$ of $x_1(n) = \{1, 2, 2, 1\}$

$x_2(n) = \{2, 1, 1, 2\}$

By Tabular Method -

range $-(N-1) \le n \le (N-1)$

	-3	-2	-1	0	1	2	3
$x_1(n)$				$\boxed{1}$	$\boxed{2}$	$\boxed{2}$	$\boxed{1}$
$x_2(n)$				2	1	1	[illegible]
$x_{2p}(n)$	1	1	2	2	1	1	[illegible]
$x_{2p}(-n)$	2	1	1	$\boxed{2}$	$\boxed{2}$	$\boxed{1}$	[illegible]
$x_2(1-n)$		2	1	1	2	2	
$x_2(2-n)$			2	1	1	2	
$x_2(3-n)$				2	1	1	

(no -ve value)

$$x_3(n) = \sum_{m=0}^{N-1} x_1(m)\, x_2((n-m))_N$$

$n = 0, 1, \ldots (N-1).$

$n=0$

$$x_3(n) = \sum_{m=0}^{3} x_1(m) \cdot x_2((-m))_N.$$

$x_3(0) = 1\times2 + 2\times2 + 2\times1\times1\times1 = 9$

$x_3(1) = 1\times1 + 2\times2 + 2\times2 + 1\times1 = 10$

$x_3(2) = 1\times1 + 2\times1 + 2\times2 + 1\times2 = 9$

$x_3(n) = 1\times2 + 2\times1 + 2\times1 + 1\times2 = 8.$

$\therefore x_3(n) = \{9, 10, 9, 8\}$

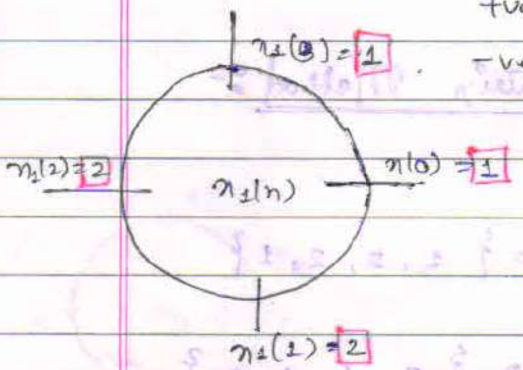
By Using Circular Array :-

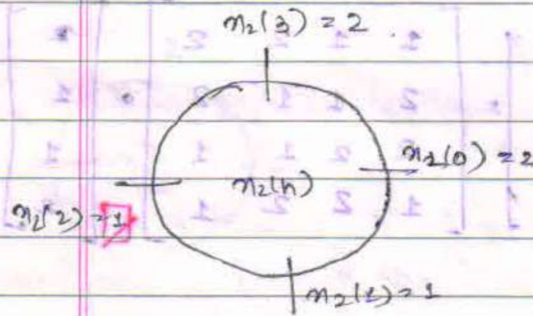
$x_1(n) = \{1, 2, 2, 1\}$

$x_2(n) = \{2, 1, 1, 2\}$

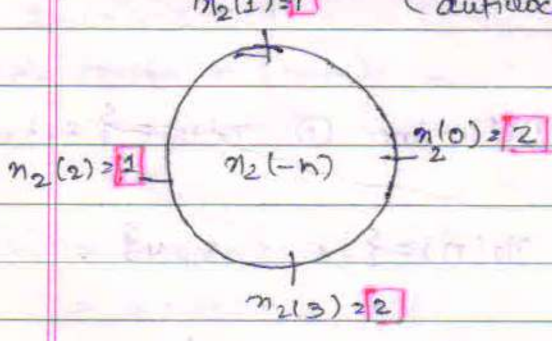
+ve → clockwise

−ve → anticlockwise



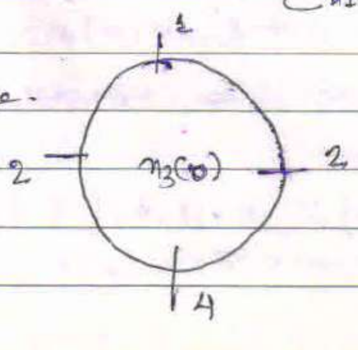


(anticlockwise)

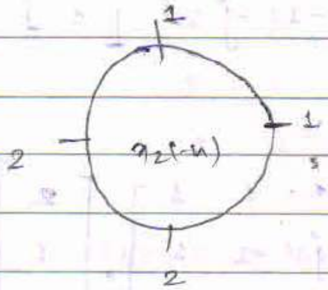


$\sum x_1(n) \cdot x_2(-n) = 9.$

product sequence.



$x_3(1) = x_1(n) +$ clockwise shift of $x_2(-n)$.



$\sum x_1(n) \cdot x_2(-n)$

$x_3(1) = 1\times1 + 1\times1 + 2\times2 + 2\times2$

$= 10$

$x_3(2)$ $x_1(n) + 2$ times clockwise shift of $x_2(-n)$

$x_3(2) = 2\times1 + 1\times1 + 1\times2 + 2\times2$

$= 9.$

$x_3(3) = x_1(n) + 3$ times clockwise shift of $x_2(-n)$

$x_3(n) = 8.$

$\therefore x_3(n) = \{9, 10, 9, 8\}$

Imp.

By Using DFT & IDFT Method :-

If $X_3(K) = X_1(K) \cdot X_2(K)$

$x_3(n) = x_1(n) \circledast x_2(n)$

$x_1(n) = \{1, 2, 2, 1\}$

$x_2(n) = \{2, 1, 1, 2\}$

$$\begin{bmatrix} x_1(0) \\ x_1(1) \\ x_1(2) \\ x_1(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_2(0) \\ x_2(1) \\ x_2(2) \\ x_2(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & 1 \\ 1 & -j & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

$X_1(0) = 1 + 2 + 2 + 1 = 6$

$X_1(1) = 1 - 2j - 2 + j = -1 - j$

$X_1(2) = \cancel{1} - \cancel{2} + \cancel{2} - j = \cancel{2} - \cancel{2}j \; 0$

$X_1(3) = 2 + j - 1 - 2j = 1 - j$

$X_2(0) = 6 \quad X_2(1) = 1 + j$

$X_2(3) = 0 \quad X_2(3) = 1 - j$

$X_1(K) = \{6, (-1-j), 0, (-1+j)\}$

$X_2(K) = \{6, (1+j), 0, (1-j)\}$

$\therefore X_3(K) = \{36, -2j, 0, 2j\}$.

$$\begin{bmatrix} x_3(0) \\ x_3(1) \\ x_3(2) \\ x_3(3) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^{-1} & W_4^{-2} & W_4^{-3} \\ 1 & W_4^{-2} & W_4^{-4} & W_4^{-6} \\ 1 & W_4^{-3} & W_4^{-6} & W_4^{-9} \end{bmatrix} \cdot \begin{bmatrix} 36 \\ -2j \\ 0 \\ 2j \end{bmatrix}$$

$$\begin{bmatrix} x_3(0) \\ x_3(1) \\ x_3(2) \\ x_3(3) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -j & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 36 \\ -2j \\ 0 \\ 2j \end{bmatrix}.$$

$x_3(n) = \{9, 10, 9, 8\}$

By Matrix Method :-

$x_1(n) = \{1, 2, 2, 1\}$

$x_2(n) = \{2, 1, 1, 2\}$

$$\begin{bmatrix} x_3(0) \\ x_3(1) \\ x_3(2) \\ x_3(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 1 & 1 & 2 \\ 2 & 2 & 1 & 1 \\ 1 & 2 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

circ shift $x_1(n)$ seq.

$x_3(n) = \{9, 10, 9, 8\}$.

Q. Determine (i) $x_1(n) = \{2, 1, 2, ...$

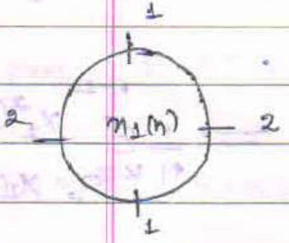
$x_2(n) = \{1, 2, 3, 4\}$.

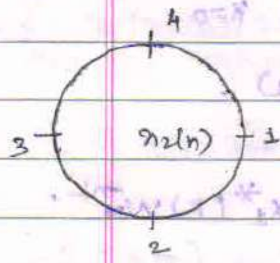
ans - $\{14, 16, 14, 16\}$

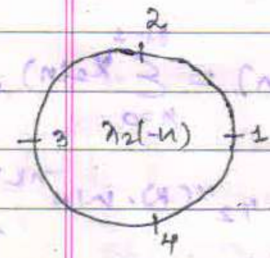
Circular array.

$x_1(n) = \{2, 1, 2, 1\}$

$x_2(n) = \{1, 2, 3, 4\}$







$x_3(0) = \sum x_1(m)\, x_2(-m)$
$= 2+6+4+2 = 14$

$x_3(1) = \sum x_1(m)\, x_2(1-m)$
$= 1+8+3+4 = 16.$

$x_3(2) = \sum x_1(m)\, x_2(2-m)$
$= 6+4+2+2 = 14$

$x_3(3) = \sum x_1(m)\, x_2(3-m)$
$= 4+3+8+1 = 16.$

$\therefore x_3(n) = \{14, 16, 14, 16\}$

Matrix Method.

$$x_3(n) = \begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} =$$

$x_3(n) = \{14, 16, 14, 16\}$

8. Complex Conjugate Property :-

If $x(n) \xleftrightarrow[DFT]{N} X(K)$

then $x^*(n) \xleftrightarrow[DFT]{N} X^*((-K))_N$

$= X^*(N-K)$

proof :- $DFT[x(n)] = \sum_{n=0}^{(N-1)} x(n) W_N^{nK}$

$K = 0, 1, \ldots, (N-1)$.

$DFT[x^*(n)] = \sum_{n=0}^{N-1} x^*(n) W_N^{nK}$

$K = 0, 1, \ldots, (N-1)$

$DFT[x^*(n)] = \sum_{n=0}^{N-1} x^*(n) W_N^{nK} W_N^{-Nn}$

$= \sum_{n=0}^{N-1} x^*(n) W_N^{-(N-K)n}.$

(multiplying $W_N^{-Nn} = 1$)

$DFT[x^*(n)] = \sum_{n=0}^{(N-1)} x^*(n) W_N^{nK} W_N^{-Nn}$

$= \sum_{n=0}^{N-1} [x(n) W_N^{(N-K)n}]^*$

$= X^*(N-K)$