# Analysis of two-hinged arch

## Introduction

Mainly three types of arches are used in practice: three-hinged, two-hinged and hingeless arches. In the early part of the nineteenth century, three-hinged arches were commonly used for the long span structures as the analysis of such arches could be done with confidence. However, with the development in structural analysis, for long span structures starting from late nineteenth century engineers adopted two-hinged and hingeless arches. Two-hinged arch is the statically indeterminate structure to degree one. Usually, the horizontal reaction is treated as the redundant and is evaluated by the method of least work. In this lesson, the analysis of two-hinged arches is discussed and few problems are solved to illustrate the procedure for calculating the internal forces.

## Analysis of two-hinged arch

A typical two-hinged arch is shown in Fig. 33.1a. In the case of two-hinged arch, we have four unknown reactions, but there are only three equations of equilibrium available. Hence, the degree of statical indeterminacy is one for two hinged arch.

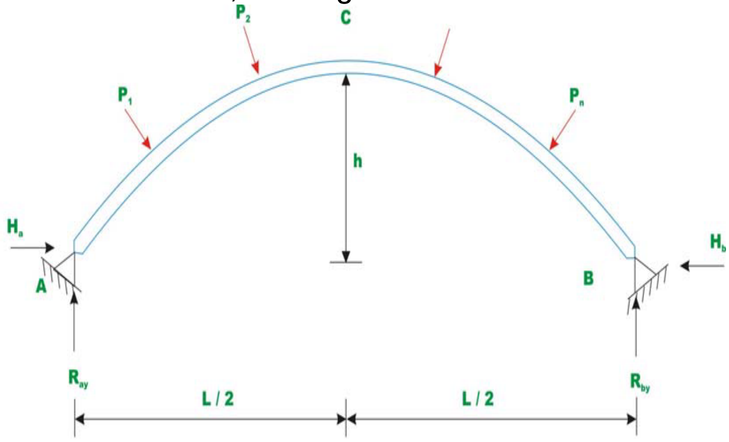


**Fig. 33.1a Two - hinged arch.**

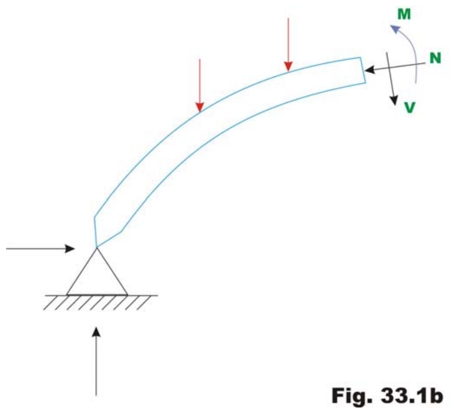


**Fig. 33.1b**

The fourth equation is written considering deformation of the arch. The unknown redundant reaction is calculated by noting that the horizontal displacement of hinge $_b$ $HB$ is zero. In general the horizontal reaction in the two hinged arch is evaluated by straightforward application of the theorem of least work (see module 1, lesson 4), which states that the partial derivative of the strain energy of a statically indeterminate structure with respect to statically indeterminate action should vanish. Hence to obtain, horizontal reaction, one must develop an expression for strain energy. Typically, any section of the arch (vide Fig 33.1b) is subjected to shear force $V$ , bending moment $M$ and the axial compression . The strain energy due to bending is calculated from the following expression.

$$U_b = \int_0^s \frac{M^2}{2EI} ds \tag{33.1}$$

The above expression is similar to the one used in the case of straight beams. However, in this case, the integration needs to be evaluated along the curved arch length. In the above equation, $s$ is the length of the centerline of the arch, $I$ is the moment of inertia of the arch cross section, $E$ is the Young's modulus of the arch material. The strain energy due to shear is small as compared to the strain energy due to bending and is usually neglected in the analysis. In the case of flat arches, the strain energy due to axial compression can be appreciable and is given by,

$$U_a = \int_0^s \frac{N^2}{2AE} ds \tag{33.2}$$

The total strain energy of the arch is given by,

$$U = \int_0^s \frac{M^2}{2EI} ds + \int_0^s \frac{N^2}{2AE} ds \tag{33.3}$$

Now, according to the principle of least work

$\frac{\partial U}{\partial H} = 0$ , where $H$ is chosen as the redundant reaction.

$$\frac{\partial U}{\partial H} = \int_0^s \frac{M}{EI} \frac{\partial M}{\partial H} ds + \int_0^s \frac{N}{AE} \frac{\partial N}{\partial H} ds = 0 \tag{33.4}$$

Solving equation 33.4, the horizontal reaction $H$ is evaluated.

## EDDY'S THEOREM

Consider a section at $P$ distant $x$ from $A$, of an arch, shown in Fig. 16·3. Let the other co-ordinate of $P$ be $y$. For the given

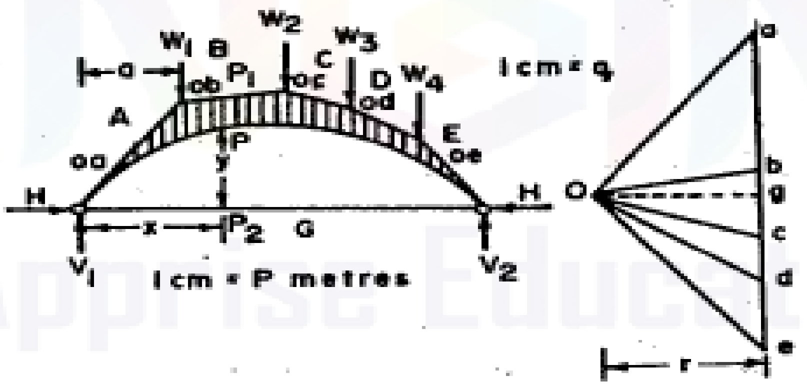


Fig. 16·3.

system of loads, the linear arch can be constructed (if $H$ is known). Since funicular polygon represents the bending moment diagram to one scale, the vertical intercept $P_1P_2$ at the section $P$ will give the bending moment due to external load system. If the arch is drawn to a scale of 1 cm $= p$ m, load diagram is plotted to a scale 1 cm $= q$ N and if the distance of pole $O$ from the load line is $r$, the scale of bending moment diagram will be 1 cm $= p.q.r.$ N-m.

Now, theoretically, the B.M. at $P$ is given by

$$M_P = -V_1x + W_1(x-a) + Hy$$
$$= \mu_X + Hy$$

where $\mu x = -V_1x + W_1(x-a)$

= Usual bending moment at a section due to load system on a simply supported beam.

From Fig. 16·3, we have,

$$\mu x = -(P_1P_2) \times \text{scale of B.M. diagram}$$
$$= -P_1P_2\,(p.q.r)$$

and $Hy = (PP_2) \times$ scale of B.M. diagram

$$= PP_2(p.q.r.)$$

Hence $M_P = \mu_X + Hy = -P_1P_2(p.q.r.) + PP_2(p.q.r.)$

$$= -(PP_1)(p.q.r.)$$

Hence the ordinate between the linear arch and the actual arch gives the bending moment. This is known as Eddy's theorem and may be stated as below :

*"The bending moment at any section of an arch is equal to the vertical intercept between the linear arch and the centre line of the actual arch".*

**Example 33.1**

A semicircular two hinged arch of constant cross section is subjected to a concentrated load as shown in Fig 33.4a. Calculate reactions of the arch and draw bending moment diagram.

**Solution:**

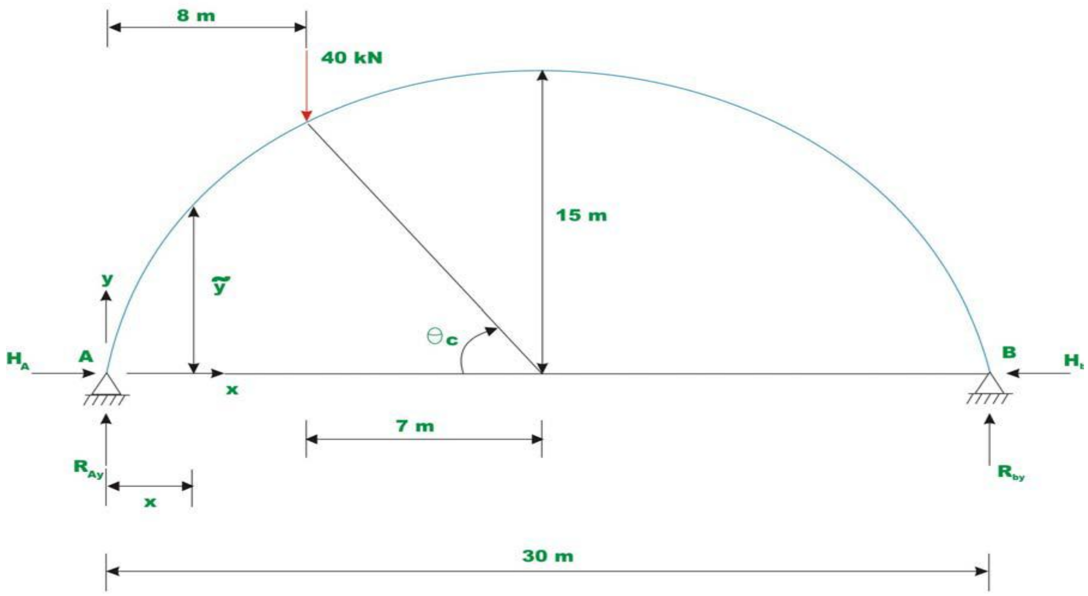


Fig. 33.4a.

Taking moment of all forces about hinge $B$ leads to,

$$R_{ay} = \frac{40 \times 22}{30} = 29.33 \text{ kN } (\uparrow)$$

$$\sum Fy = 0 \qquad \Rightarrow R_{by} = 10.67 \text{ kN } (\uparrow) \qquad (1)$$

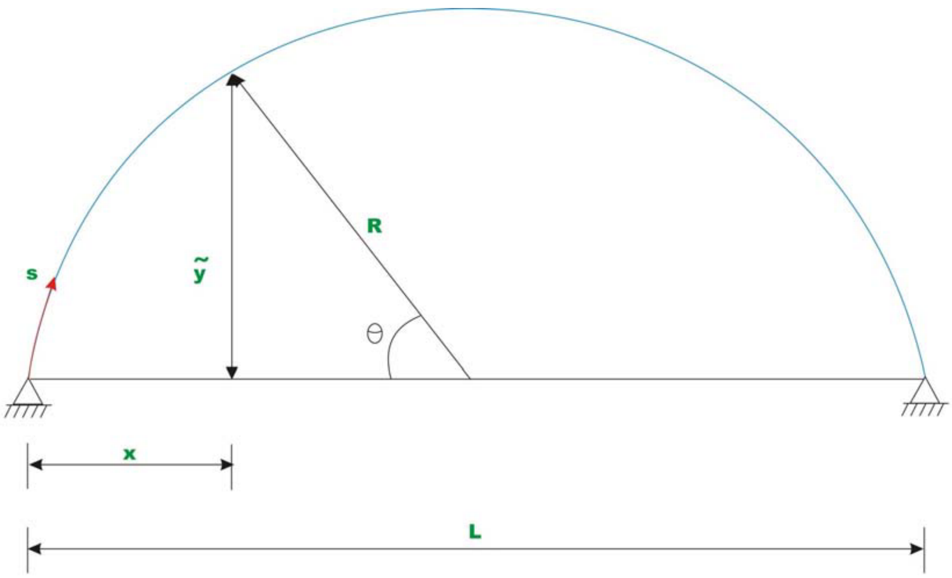


Fig. 33.4b.

From Fig. 33.4b,

$$\tilde{y} = R\sin\theta$$

$$x = R(1-\cos\theta)$$

$$ds = R\,d\theta \tag{2}$$

$$\tan\theta_c = \frac{13.267}{7} \qquad \Rightarrow \theta_c = 62.18^\circ = \pi/_{2.895}\,rad$$

Now, the horizontal reaction $H$ may be calculated by the following expression,

$$H = \frac{\int_0^s M_0 \tilde{y}\, ds}{\int_0^s \tilde{y}^2\, ds} \tag{3}$$

Now $M_0$ the bending moment at any cross section of the arch when one of the hinges is replaced by a roller support is given by,

$$M_0 = R_{ay}x = R_{ay}\,R(1-\cos\theta) \qquad\qquad 0 \le \theta \le \theta_c$$

and,

$$\begin{aligned} M_0 &= R_{ay}\,R(1-\cos\theta) - 40(x-8) \\ &= R_{ay}\,R(1-\cos\theta) - 40\{R(1-\cos\theta)-8\} \qquad \theta_c \le \theta \le \pi \end{aligned} \tag{4}$$

Integrating the numerator in equation (3),

$$\int_0^s M_0\tilde{y}ds = \int_0^{\theta_c} R_{ay}R^3(1-\cos\theta)\sin\theta\,d\theta + \int_{\theta_c}^{\pi}[R_{ay}\,R(1-\cos\theta) - 40\{R(1-\cos\theta)-8\}]R\sin\theta\,Rd\theta$$

$$= R_{ay}R^3\int_0^{\pi/2.895}(1-\cos\theta)\sin\theta\,d\theta + R^2\int_{\pi/2.895}^{\pi}[R_{ay}\,R(1-\cos\theta)\sin\theta - 40\{R(1-\cos\theta)\sin\theta - 8\sin\theta\}]d\theta$$

$$= R_{ay}R^3[-\cos\theta]_0^{\pi/2.895} + R^2\left[\left[R_{ay}\,R(-\cos\theta)\right]_{\pi/2.895}^{\pi} - \left[40\,R(-\cos\theta)\right]_{\pi/2.895}^{\pi} + \left[40\times 8(-\cos\theta)\right]_{\pi/2.895}^{\pi}\right]$$

$$= 0.533R_{ay}R^3 + R^2[[1.4667R_{ay}\,R] - [40\,R(1.4667)] + [40\times 8(1.4667)]]$$

$$= 52761.00 + 225(645.275 - 410.676) = 105545.775 \tag{5}$$

The value of denominator in equation (3), after integration is,

$$\begin{aligned} \int_0^s \tilde{y}^2 ds &= \int_0^{\pi}(R\sin\theta)^2 Rd\theta \\ &= R^3\int_0^{\pi}\left(\frac{1-\cos 2\theta}{2}\right)d\theta = R^3\left(\frac{\pi}{2}\right) = 5301.46 \end{aligned} \tag{6}$$

Hence, the horizontal thrust at the support is,

$$H = \frac{105545.775}{5301.46} = 19.90 \quad \text{kN} \tag{7}$$

Bending moment diagram

Bending moment $M$ at any cross section of the arch is given by,

$$M = M_0 - H\bar{y}$$
$$= R_{ay}R(1-\cos\theta) - HR\sin\theta \qquad 0 \le \theta \le \theta_c \qquad (8)$$
$$= 439.95(1-\cos\theta) - 298.5\sin\theta$$

$$M = 439.95(1-\cos\theta) - 298.5\sin\theta - 40(15(1-\cos\theta)-8) \qquad \theta_c \le \theta \le \pi \qquad (9)$$

Using equations (8) and (9), bending moment at any angle $\theta$ can be computed. The bending moment diagram is shown in Fig. 33.4c.

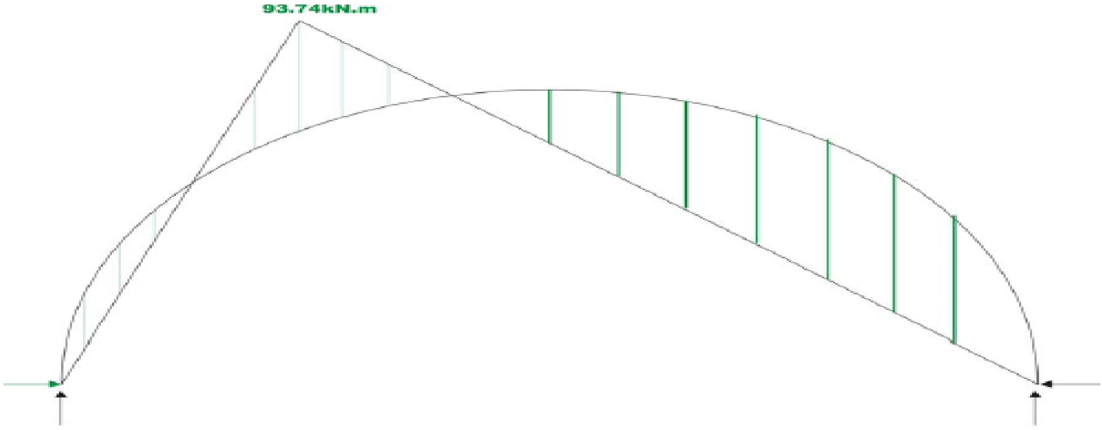


**Fig. 33.4c Bending moment diagram**

## Example 33.2

A two hinged parabolic arch of constant cross section has a span of 60m and a rise of 10m. It is subjected to loading as shown in Fig.33.5a. Calculate reactions of the arch if the temperature of the arch is raised by $40^\circ\ C$. Assume co-efficient of thermal expansion as $\alpha = 12 \times 10^{-6} / ^\circ C$.

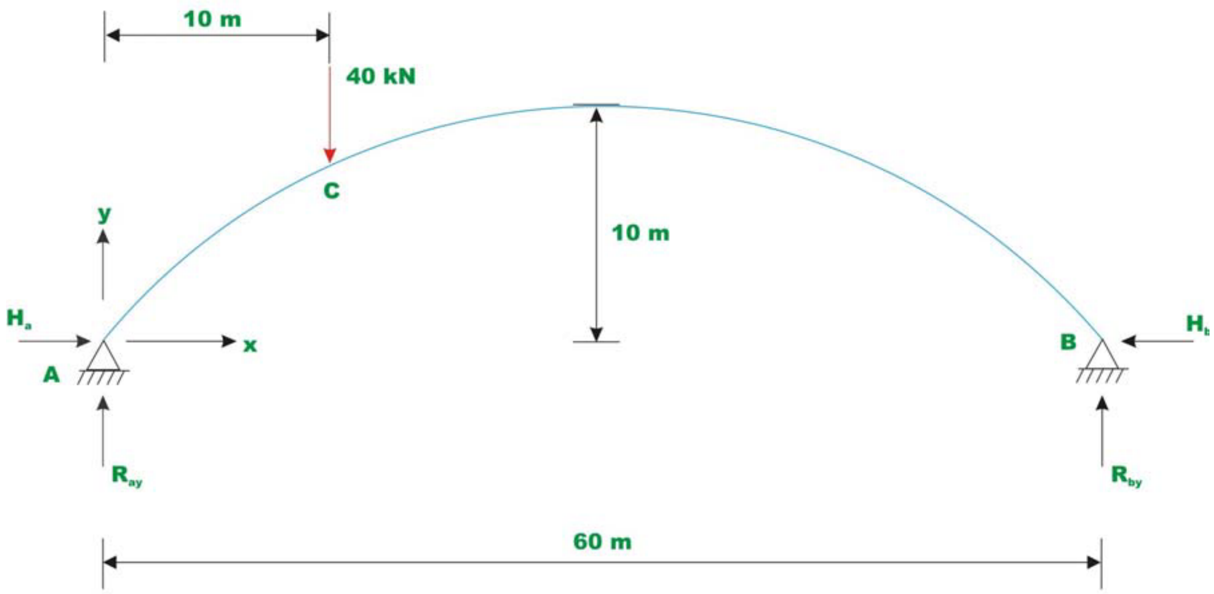


**Fig. 33.5**

Taking $A$ as the origin, the equation of two hinged parabolic arch may be written as,

$$y = \frac{2}{3}x - \frac{10}{30^2}x^2 \qquad (1)$$

The given problem is solved in two steps. In the first step calculate the horizontal reaction due to load 40 kN applied at . In the next step calculate the horizontal reaction due to rise in temperature. Adding both, one gets the horizontal reaction at the hinges due to combined external loading and temperature change. The horizontal reaction due to 40 kN load may be calculated by the following equation,

$$H_1 = \frac{\int_0^s M_0 y\, ds}{\int_0^s \bar{y}^2\, ds} \qquad (2a)$$

For temperature loading, horizontal reaction is given by,

$$H_2 = \frac{\alpha L T}{\int_0^s \frac{y^2}{EI} ds} \qquad (2b)$$

Where $L$ is the span of the arch.

For 40 kN load,

$$\int_0^s M_0 y\, ds = \int_0^{10} R_{ay}\, x y\, dx + \int_{10}^{60} [R_{ay} x - 40(x-10)] y\, dx \qquad (3)$$

Please note that in the above equation, the integrations are carried out along the $x$-axis instead of the curved arch axis. The error introduced by this change in the variables in the case of flat arches is negligible. Using equation (1), the above equation (3) can be easily evaluated.

The vertical reaction $A$ is calculated by taking moment of all forces about $B$. Hence,

$$R_{ay} = \frac{1}{60}[40 \times 50] = 33.33 \quad \text{kN}$$

$$R_{by} = 6.67 \quad \text{kN}.$$

Now consider the equation (3),

$$\int_0^l M_0 y\, dx = \int_0^{10} (33.33)\, x (\frac{2}{3}x - \frac{10}{30^2}x^2)\, dx + \int_{10}^{60} [(33.33)x - 40(x-10)](\frac{2}{3}x - \frac{10}{30^2}x^2)\, dx$$

$$= 6480.76 + 69404.99 = 74885.75 \qquad (4)$$

$$\int_0^l y^2\, dx = \int_0^{60} \left[\frac{2}{3}x - \frac{10}{30^2}x^2\right]^2 dx$$

$$= 3200 \qquad (5)$$

Hence, the horizontal reaction due to applied mechanical loads alone is given by,

$$H_1 = \frac{\int_0^l M_0 y \quad dx}{\int_0^l y^2 \quad dx} = \frac{75885.75}{3200} = 23.71 \quad \text{kN} \qquad (6)$$

The horizontal reaction due to rise in temperature is calculated by equation (2b),

$$H_2 = \frac{12 \times 10^{-6} \times 60 \times 40}{3200 / EI} = \frac{EI \times 12 \times 10^{-6} \times 60 \times 40}{3200}$$

Taking $E = 200 \quad \text{kN/mm}^2$ and $I = 0.0333 m^4$

$$H_2 = 59.94 \quad \text{kN}. \qquad (7)$$

Hence the total horizontal thrust $H = H_1 + H_2 = 83.65$ kN.

When the arch shape is more complicated, the integrations $\int_0^s \frac{M_0 y}{EI} ds$ and $\int_0^s \frac{y^2}{EI} ds$ are accomplished numerically. For this purpose, divide the arch span in to $n$ equals divisions. Length of each division is represented by $(\Delta s)_i$ (vide Fig.33.5b). At the midpoint of each division calculate the ordinate $y_i$ by using the equation $y = \frac{2}{3}x - \frac{10}{30^2}x^2$. The above integrals are approximated as,

$$\int_0^s \frac{M_0 y}{EI} ds = \frac{1}{EI}\sum_{i=1}^{n}(M_0)_i y_i (\Delta s)_i \tag{8}$$

$$\int_0^s \frac{y^2}{EI} ds = \frac{1}{EI}\sum_{i=1}^{n}(y)_i^2 (\Delta s)_i \tag{9}$$

The complete computation for the above problem for the case of external loading is shown in the following table.

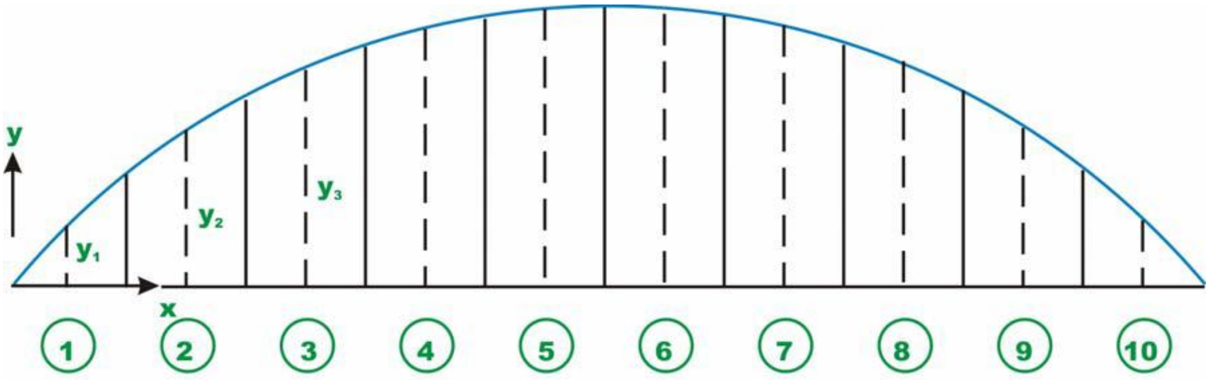


**Fig. 33.5(b)**

Table 1. Numerical integration of equations (8) and (9)

| Segment No | Horizontal distance $x$ Measured from A (m) | Corresponding $y_i$ (m) | Moment at that Point $(M_0)_i$ (kNm) | $(M_0)_i y_i (\Delta s)_i$ | $(y)_i^2 (\Delta s)_i$ |
|---|---|---|---|---|---|
| 1 | 3 | 1.9 | 99.99 | 1139.886 | 21.66 |
| 2 | 9 | 5.1 | 299.97 | 9179.082 | 156.06 |
| 3 | 15 | 7.5 | 299.95 | 13497.75 | 337.5 |
| 4 | 21 | 9.1 | 259.93 | 14192.18 | 496.86 |
| 5 | 27 | 9.9 | 219.91 | 13062.65 | 588.06 |
| 6 | 33 | 9.9 | 179.89 | 10685.47 | 588.06 |
| 7 | 39 | 9.1 | 139.87 | 7636.902 | 496.86 |
| 8 | 45 | 7.5 | 99.85 | 4493.25 | 337.5 |
| 9 | 51 | 5.1 | 59.83 | 1830.798 | 156.06 |
| 10 | 57 | 1.9 | 19.81 | 225.834 | 21.66 |
| | | | $\sum$ | 75943.8 | 3300.3 |

$$H_1 = \frac{\sum (M_0)_i y_i (\Delta s)}{\sum (y)_i^2 (\Delta s)_i} = \frac{75943.8}{3200.3} = 23.73 \text{ kN} \tag{10}$$

This compares well with the horizontal reaction computed from the exact integration.

## Summary

Two-hinged arch is the statically indeterminate structure to degree one. Usually, the horizontal reaction is treated as the redundant and is evaluated by the method of least work. Towards this end, the strain energy stored in the two hinged arch during deformation is given. The reactions developed due to thermal loadings are discussed. Finally, a few numerical examples are solved to illustrate the procedure.