Lecture - 7

Surface Integral - Any integral which is to be evaluated over a surface is called surface integral.

$$\text{Surface Integral} = \iint_S \bar{F} \cdot \hat{n} \, ds$$

where $\bar{F}$ be a vector point function and S be the given surface

$\hat{n}$ is a unit normal vector

$\hat{n} = \dfrac{\bar{n}}{|\bar{n}|}$ and $\bar{n} = \text{grad}\, \phi$

Note:-

Projection

On xy-plane then $ds = \dfrac{dx\,dy}{|\hat{n} \cdot \hat{k}|}$

On yz-plane then $ds = \dfrac{dy\,dz}{|\hat{n} \cdot \hat{i}|}$

On zx-plane then $ds = \dfrac{dz\,dx}{|\hat{n} \cdot \hat{j}|}$

RGPV Dec-2014

Q.1) Evaluate $\iint_S \bar{A}\cdot\hat{n}\, ds$ where $\bar{A} = 18z\hat{i} - 12\hat{j} + 3y\hat{k}$ and S is the part of the plane $2x+3y+6z=12$ in the first octant.

Solution:- Surface Integral $= \iint_S \bar{A}\cdot\hat{n}\, dS$ — (1)

We have $\bar{A} = 18z\hat{i} - 12\hat{j} + 3y\hat{k}$

and $\phi = 2x+3y+6z-12$

Now grad $\phi = \nabla\phi$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(2x+3y+6z-12)$$

$$= \hat{i}\frac{\partial}{\partial x}(2x+3y+6z-12) + \hat{j}\frac{\partial}{\partial y}(2x+3y+6z-12) + \hat{k}\frac{\partial}{\partial y}(2x+3y+6z-12)$$

$$= \hat{i}(2) + \hat{j}(3) + \hat{k}(6)$$

$$\text{grad}\,\phi = 2\hat{i} + 3\hat{j} + 6\hat{k}$$

$$|\text{grad}\,\phi| = \sqrt{(2)^2 + (3)^2 + (6)^2}$$

$$= \sqrt{4+9+36}$$

$$= \sqrt{49} = 7$$

$$\hat{n} = \frac{\text{grad}\,\phi}{|\text{grad}\,\phi|} = \frac{2\hat{i}+3\hat{j}+6\hat{k}}{7}$$

let the Projection of S on the xy-plane

then $ds = \dfrac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$

$$= \frac{dx\,dy}{\left|\frac{(2\hat{i}+3\hat{j}+6\hat{k})}{7}\cdot\hat{k}\right|}$$

$$= \frac{dx\,dy}{\left(\frac{6}{7}\right)}$$

$$ds = \frac{7}{6}\,dx\,dy$$

and $\bar{A}\cdot\hat{n} = (18z\hat{i}-12\hat{j}+3y\hat{k})\cdot\left(\dfrac{2\hat{i}+3\hat{j}+6\hat{k}}{7}\right)$

$$\bar{A}\cdot\hat{n} = \frac{36z-36+18y}{7}$$

From equation ①

$$S.I = \iint_S \left(\frac{36z-36+18y}{\cancel{7}}\right)\frac{\cancel{7}}{6}\,dx\,dy$$

$$S.I. = \iint_S (6z-6+3y)\,dx\,dy \quad — (11)$$

$\because$ $2x+3y+6z=12$

$6z = 12-2x-3y$

$$= \iint_S \left(12 - 2x - \cancel{3y} - 6 + \cancel{3y}\right) dx\,dy$$

$$= \iint_S (6 - 2x)\, dx\,dy \quad - \quad (\text{iii})$$

Now find limit from $2x + 3y + 6z = 12$

Put $z = 0$

$$2x + 3y = 12 \quad - \quad (\text{iv})$$

$$3y = 12 - 2x$$

$$y = \frac{12 - 2x}{3}$$

Limit of y : $0 \rightarrow \frac{12-2x}{3}$

Put $y = 0$ in (iv)

$$2x = 12 \quad \therefore \quad x = 6$$

Limit of x: $0 \rightarrow 6$

From (iii)

$$SI = \int_{x=0}^{x=6} \int_{y=0}^{y=\frac{12-2x}{3}} (6 - 2x)\, dx\,dy$$

First integrate w. r. to y

$$= \int_0^6 (6 - 2x) \left[\int_0^{\frac{12-2x}{3}} 1 \, dy \right] dx$$

$$= \int_0^6 (6-2x)\left[y\right]_0^{\frac{12-2x}{3}} dx$$

$$= \int_0^6 (6-2x)\left(\frac{12-2x}{3}\right) dx$$

$$= \int_0^6 \frac{2(3-x)\,2(6-x)}{3}\, dx$$

$$= \frac{4}{3}\int_0^6 (3-x)(6-x)\, dx$$

$$= \frac{4}{3}\int_0^6 (18-3x-6x+x^2)\, dx$$

$$= \frac{4}{3}\int_0^6 (x^2-9x+18)\, dx$$

$$= \frac{4}{3}\left[\frac{x^3}{3}-\frac{9x^2}{2}+18x\right]_0^6$$

$$= \frac{4}{3}\left[\frac{(6)^3}{3}-\frac{9(6)^2}{2}+18(6)\right]$$

$$= \frac{4}{3}\left[\cancel{\frac{216}{3}}^{72}-\cancel{\frac{324}{2}}^{162}+108\right]$$

$$= \frac{4}{3}\left[72-162+108\right]$$

$$= \frac{4}{\cancel{3}}\left[\cancel{18}^{6}\right] = 24 \quad \text{Ans.}$$

Q.2) RGPV December - 2015

Evaluate $\iint_S \bar{A}\cdot\hat{n}\, ds$ where $\bar{A} = z\hat{i} + x\hat{j} - 3y^2 z\hat{k}$ and S is the surface of the cylinder $x^2+y^2=16$ included in the first octant between $z=0$ and $z=5$.

Solution:- Surface integral $= \iint_S \bar{A}\cdot\hat{n}\, ds$ — (1)

We have $\bar{A} = z\hat{i} + x\hat{j} - 3y^2 z\hat{k}$

and $\phi = x^2+y^2-16$

Now $\text{grad}\,\phi = \nabla\phi$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2-16)$$

$$= \hat{i}\frac{\partial}{\partial x}(x^2+y^2-16) + \hat{j}\frac{\partial}{\partial y}(x^2+y^2-16) + \hat{k}\frac{\partial}{\partial z}(x^2+y^2-16)$$

$$= \hat{i}(2x) + \hat{j}(2y) + 0$$

$$\text{grad}\,\phi = 2x\hat{i} + 2y\hat{j}$$

$$|\text{grad}\,\phi| = \sqrt{(2x)^2+(2y)^2}$$

$$= \sqrt{4x^2+4y^2}$$

$$|\text{grad}\,\phi| = 2\sqrt{x^2+y^2}$$

$$\hat{n} = \frac{\text{grad}\,\phi}{|\text{grad}\,\phi|} = \frac{2x\hat{i}+2y\hat{j}}{2\sqrt{x^2+y^2}} = \frac{x\hat{i}+y\hat{j}}{\sqrt{16}} = \frac{x\hat{i}+y\hat{j}}{4}.$$

let the Projection of s on the ~~zx~~ yz-plane

then $ds = \dfrac{dy\,dz}{|\hat{n}\cdot\hat{i}|}$

$$= \frac{dz\,dx}{\left|\left(\frac{x\hat{i}+y\hat{j}}{4}\right)\cdot\hat{i}\right|}$$

$$ds = \frac{dy\,dz}{\frac{x}{4}} = \frac{4}{x}\,dy\,dz$$

and $\bar{A}\cdot\hat{n} = (z\hat{i}+x\hat{j}-3y^2z\hat{k})\cdot\dfrac{(x\hat{i}+y\hat{j})}{4}$

$$= \frac{zx+xy}{4}$$

from eqn. (1)

$$S.I = \iint_S \left(\frac{zx+xy}{4}\right)\frac{4}{x}\,dz\,dy$$

$$= \iint_S \frac{\cancel{x}(z+y)}{\cancel{4}}\cdot\frac{\cancel{4}}{\cancel{x}}\,dz\,dy$$

$$= \int_{y=0}^{y=4}\int_{z=0}^{z=5}(z+y)\,dz\,dy$$

$$= \int_0^4\left(\frac{z^2}{2}+yz\right)_0^5 dy = \int_0^4\left(\frac{25}{2}+5y\right)dy$$

$$= \left(\frac{25}{2}y+\frac{5y^2}{2}\right)_0^4 = \frac{100}{2}+\frac{80}{2} = 90$$

RGPV - Dec. 2016

Q.3) Evaluate $\iint_S \bar{A}\cdot\hat{n}\,ds$, where $\bar{A} = (x+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}$ and S is the surface of the plane $2x+y+2z=6$ in the first octant.

Sol. Surface integral $= \iint_S \bar{A}\cdot\hat{n}\,ds$ — (1)

We have $\bar{A} = (x+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}$

and $\phi = 2x+y+2z-6$

Now $\text{grad}\,\phi = \nabla\phi$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(2x+y+2z-6)$$

$$= \hat{i}(2) + \hat{j}(1) + \hat{k}(2)$$

$$\text{grad}\,\phi = 2\hat{i} + \hat{j} + 2\hat{k}$$

$$|\text{grad}\,\phi| = \sqrt{(2)^2+(1)^2+(2)^2}$$

$$= \sqrt{4+1+4}$$

$$= \sqrt{9} = 3$$

$$\hat{n} = \frac{\text{grad}\,\phi}{|\text{grad}\,\phi|}$$

$$\hat{n} = \frac{2\hat{i}+\hat{j}+2\hat{k}}{3}$$

let the projection of S on the xy-plane

then $ds = dx\,dy / |\hat{n}\cdot\hat{k}|$

$$ds = \frac{dx\,dy}{\left|\frac{(2\hat{i}+\hat{j}+2\hat{k})}{3}\cdot\hat{k}\right|}$$

$$ds = \frac{dx\,dy}{\left|\frac{2}{3}\right|} = \frac{3}{2}\,dx\,dy$$

and $\bar{A}\cdot\hat{n} = \left[(x+y^2)\,\hat{i} - 2x\hat{j} + 2yz\hat{k}\right]\cdot\left[\frac{2\hat{i}+\hat{j}+2\hat{k}}{3}\right]$

$$= \frac{2(x+y^2) - 2x + 4yz}{3}$$

$$= \frac{\cancel{2x} + 2y^2 - \cancel{2x} + 4yz}{3}$$

$$= \frac{2y^2 + 4yz}{3}$$

$$\bar{A}\cdot\hat{n} = \frac{2}{3}(y^2 + 2yz)$$

From eqn. 1

$$S.I = \iint_S \frac{\cancel{2}}{\cancel{3}}(y^2+2yz)\,\frac{\cancel{3}}{\cancel{2}}\,dx\,dy$$

$$= \iint_S (y^2+2yz)\,dx\,dy$$

Find the value of z from $2x+y+2z=6$

$2z = 6-y-2x$

$\therefore\ S.I = \iint_S y^2 + y(6-y-2x)\,dx\,dy$

$$S.I = \iint_S (\cancel{y^2} + 6y - \cancel{y^2} - 2xy)\, dx\,dy$$

$$= \iint_S (6y - 2xy)\, dx\,dy$$

$$= 2\iint_S (3y - xy)\, dx\,dy$$

Now find the limit from

$$2x + y + 2z = 6$$

Put $z = 0$

$$2x + y = 6 \quad — \quad (11)$$

$$y = 6 - 2x$$

Limit of y: $0 \rightarrow 6 - 2x$

Put $y = 0$ in (11)

$$2x = 6$$

$$x = 3$$

Limit of x: $0 \rightarrow 3$

Now

$$S.I = 2\int_{x=0}^{x=3} \int_{y=0}^{y=6-2x} (3y - xy)\, dx\,dy$$

First integrate w. r. to y

$$= 2\int_0^3 \left(\frac{3y^2}{2} - \frac{xy^2}{2}\right)_0^{6-2x} dx$$

$$= 2\int_0^3 \left\{ \frac{3}{2}(6-2x)^2 - \frac{x(6-2x)^2}{2} \right\} dx$$

$$= \frac{2}{2}\int_0^3 \left\{ 3(6-2x)^2 - x(6-2x)^2 \right\} dx$$

$$= \int_0^3 \underbrace{(6-2x)^2}_{v}\ \underbrace{(3-x)}_{u}\ dx$$

$$\left[\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \cdots \right]$$

$$= \left\{ (3-x)\frac{(6-2x)^3}{3(-2)} - (-1)\frac{(6-2x)^4}{4(-2)3(-2)} \right\}_0^3$$

$$= \left\{ 0 - 0 - 3\frac{(6)^3}{-6} - \frac{(6)^4}{(-8)(-6)} \right\}$$

$$= \frac{3(6)(6)\cancel{(6)}}{\cancel{6}} - \frac{(6)(6)(6)\cancel{(6)}}{8\times\cancel{6}}$$

$$= 36\left(3 - \frac{6}{8}\right)$$

$$= 36\left(\frac{24-6}{8}\right)$$

$$= \frac{{}^{9}\cancel{36}\times\cancel{18}^{\,9}}{\cancel{8}_{\,2}}$$

$$= 81$$

For Practice

Dec-2011, Dec-2013

Q.4) Evaluate $\int_S \bar{F} \cdot d\bar{S}$ where $\bar{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ and S is the surface bounding the region $x^2+y^2=4$, $z=0$ and $z=3$

Q.5) Evaluate $\iint_S \bar{F} \cdot \hat{n}\, ds$, where $\bar{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$ and S is that part of the surface of the ~~plane~~ sphere $x^2+y^2+z^2=1$ which lies in the first octant.